On the Curve of Minimal Length between Two Points On a Bézier Surface

A. Carriazo
Dpto. Geometría y Topología
Universidad de Sevilla
Sevilla, España
carriazo@us.es

M. C. Márquez
Dpto. Geometría y Topología
Universidad de Sevilla
Sevilla, España
cmgarcia@us.es

H. Ugail
Centre for Visual Computing
University of Bradford
Bradford, United Kingdom
h.ugail@bradford.ac.uk

***Abstract*—In this paper, we present a method to obtain the distance between two fixed points of a Bézier surface by minimizing the length of curves on the surface linking the two points. We use a discretization of the classical orthogonal variations method for a regular planar curve. We provide interesting examples of Bézier surfaces which approximate the cylinder, the sphere and the hyperbolic paraboloid.**

***Index Terms*—Bézier surface, geodesic curve, minimal distance**

INTRODUCTION

In the plane, in the Euclidean space, the segment between two points is the shortest curve linking them, and it gives the distance between both the points. However, what happens if we consider two points on an arbitrary surface is an intriguing question. We know that the shortest curve embedded on the surface linking two points is a geodesic and it provides the distance between the points on the surface [2]. The solution of this problem has practical applications in areas of geometric design, e.g. [3], [4], [5], [6], [7], [8], [9], [10].

Computation of geodesics on parametric surfaces can be a complicated process. A geodesic curve is the solution of the following system of equations such that,

$$\frac{d^2x_k}{ds^2} + \sum_{i,j=1}^{n} \Gamma_{ij}^k \frac{dx_i}{ds}\frac{dx_j}{ds} = 0,$$

where $x_i(s)$ are the parametric equations of the curve and Γ_{ij}^k are the Christoffel symbols of the surface. Moreover, from the calculus of variations, a geodesic curve joining two points is the curve that minimizes the squared length function, as defined in Equation (2), from all the curves on the surface linking the two points.

In order to simplify the computation of geodesics, we introduce a new method which adapts the classical orthogonal calculus of variations of curves of differential geometry for discrete orthogonal variations of Bézier curves. The advantage of our method is that the approximated geodesic curve is obtained by solving a system of polynomial equations.

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The rest of the paper is divided into four sections. The first two sections that follow introduce the notations and the problem, respectively. In the third section, we describe our method in step by step. In Section IV, we illustrate our method using three particular surfaces, namely, the cylinder, the sphere and the hyperbolic paraboloid. Finally, in Section V, we conclude the paper.

I. NOTATIONS

Let $B_i^n(u)$ and $B_j^m(v)$ be the Bernstein basis functions of degree n and m, respectively. A regular Bézier surface with the control points, $P_{ij} \in \mathbb{R}^3$ $(0 \leq i \leq n,\ 0 \leq j \leq m)$ is the parametric surface defined by,

$$\chi(u,v) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{ij} B_i^n(u) B_j^m(v), \qquad (1)$$

for $(u,v) \in U = [0,1]\times[0,1] \subset \mathbb{R}^2$, such that $\frac{\partial\chi}{\partial u}\times\frac{\partial\chi}{\partial v}(u,v) \neq 0$, for $(u,v) \in U$.

Given Q_0 and Q_1 defined to be two fixed points on $\chi(u,v)$, there are two fixed points q_0 and q_1 in U such that $\chi(q_0) = Q_0$ and $\chi(q_1) = Q_1$.

A Bézier curve of degree l in U linking q_0 and q_1 is determined by a set of control points in U, $\{P_i\}_{i=1}^{l-1}$, and has the following parametric equations forms such that,

$$\alpha(t) = B_0^l(t)q_0 + \sum_{i=1}^{l-1} B_i^l(t)P_i + B_l^l(t)q_1,$$

for $t \in [0,1]$.

The parametrization χ, maps the Bézier curve $\alpha(t)$ to a curve on the Bézier surface, $\beta(t) = \chi(\alpha(t))$, which links $Q_0 = \beta(0)$ and $Q_1 = \beta(1)$. Moreover, $\alpha(t) = (u(t), v(t))$ with,

$$u(t) = B_0^l(t)q_0^1 + \sum_{i=1}^{l-1} B_i^l(t)P_i^1 + B_l^l q_1^1,$$

$$v(t) = B_0^l(t)q_0^2 + \sum_{i=1}^{l-1} B_i^l(t)P_i^2 + B_l^l q_1^2,$$

where $q_0 = (q_0^1, q_0^2)$, $q_1 = (q_1^1, q_1^2)$, $P_i = (P_i^1, P_i^2)$ are the coordinates of these points. We note that if we fix the control points of the Bézier curve $\alpha(t)$ and those of the Bézier surface $\chi(u,v)$, then $u(t)$ and $v(t)$ are polynomials of degree l in the variable t and the curve $\beta(t) = \chi(u(t), v(t))$ is also a polynomial curve in $\mathbb{R}^3$. In particular,

$$\beta(t) = \sum_{i=0}^{n}\sum_{j=0}^{m} P_{ij} B_i^n(u(t)) B_j^m(v(t)),$$

has polynomials of degree $n \cdot m \cdot l^2$ in the variable t in its three coordinates.

The length of the curve $\beta(t)$ is defined by $L(\beta) = \int_0^1 \sqrt{|\beta'(t)|}dt$, where $|\ \ |$ denotes the norm of vectors in the Euclidean space $\mathbb{R}^3$. We can consider the squared length defined by,

$$L^2(\beta) = \int_0^1 |\beta'(t)|dt, \tag{2}$$

which is easier to compute.

II. The Problem

Given a parametric surface and two points on it, we want to find a geodesic curve on the surface joining these points.

In order to provide an approximate solution of this problem, we first propose to approach the surface by a Bézier surface. There can be several methods to produce such an approximation. We propose the one given in [1] by the authors. By using that normal approximation method, every parametric surface can be approximated by a unique Bézier surface. Next, by following the method that we introduce in this paper, we approach the geodesic by a polynomial curve.

III. The Method

In this section, we propose a method to approximate the distance between two points Q_0 and Q_1 on the Bézier surface $\chi(u,v)$, by minimizing the length of curves linking these two points. Therefore, by fixing a regular Bézier surface $\chi(u,v)$ and two points Q_0 and Q_1 on it, we obtain a polynomial curve linking both points which provides us an approximation of the distance between Q_0 and Q_1 on $\chi(u,v)$.

To describe this method we shall follow the following steps.

Step 1: We fix q_0 and q_1 in $U = [0,1] \times [0,1] \subset \mathbb{R}^2$ such that $\chi(q_0) = Q_0$ and $\chi(q_1) = Q_1$ and we describe the discrete orthogonal variations of a regular parametrized plane curve linking the points q_0 and q_1 in $U \subset \mathbb{R}^2$.

We consider $\alpha : [0,1] \longrightarrow U \subset \mathbb{R}^2$ a regular parametrized plane curve in U such that $\alpha(0) = q_0$ and $\alpha(1) = q_1$ and we denote by $\mathbf{n}(t)$ the unit normal vector to the curve α at the point $\alpha(t)$. We choose $n+1$ values $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ and we obtain $n+1$ points of the curve $P_i = \alpha(t_i)$. For each $i \in \{1, \ldots, n-1\}$ we consider the points $P_i(\lambda_i) = P_i + \lambda_i \mathbf{n}(t_i)$, with $\lambda_i \in \mathbb{R}$ such that $P_i(\lambda_i) \in U$.

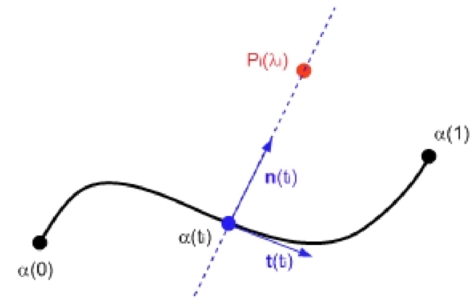


Figure 0. From all these points, we construct a family of Bézier curves.

$$\alpha(\lambda_1, \ldots, \lambda_{n-1}, t) = B_0^n(t) q_0 + \sum_{i=1}^{n-1} B_i^n(t) P_i(\lambda_i) + B_n^n(t) q_1,$$

depending on $\lambda_1, \ldots, \lambda_{n-1}$, all of them parametrized for $t \in [0,1]$ and satisfying that $\alpha(\lambda_1, \ldots, \lambda_{n-1}, 0) = q_0$ and $\alpha(\lambda_1, \ldots, \lambda_{n-1}, 1) = q_1$.

Step 2: The parametrization χ defined in (1) maps this family to the following family of polynomial curves on the Bézier surface $\chi(u,v)$:

$$\beta(\lambda_1, \ldots, \lambda_{n-1}, t) = \chi(\alpha(\lambda_1, \ldots, \lambda_{n-1}, t)),$$

depending on $\lambda_1, \ldots, \lambda_{n-1}$, all the curves parametrized by $t \in [0,1]$ and satisfying that $\beta(\lambda_1, \ldots, \lambda_{n-1}, 0) = \chi(q_0) = Q_0$ and $\beta(\lambda_1, \ldots, \lambda_{n-1}, 1) = \chi(q_1) = Q_1$.

Step 3: We define the function,

$$F(\lambda_1, \ldots, \lambda_{n-1}) = \int_0^1 |\frac{d}{dt}\beta(\lambda_1, \ldots, \lambda_{n-1}, t)|^2 dt,$$

which represents the squared length of the curve $\beta(\lambda_1, \ldots, \lambda_{n-1}, t)$. Note that as a consequence of the fact that the curve $\beta(\lambda_1, \ldots, \lambda_{n-1}, t)$ is polynomial, we have that the function $F(\lambda_1, \ldots, \lambda_{n-1})$ is also polynomial.

Step 4: We want to minimize the function, $F(\lambda_1, \ldots, \lambda_{n-1})$ and to obtain a curve whose length is minimal from all the curves of the family. We need to solve the following polynomial equations system,

$$\frac{\partial F}{\partial \lambda_i} = 0, \quad \text{for all } \ i = 1, \ldots n-1.$$

A solution $\lambda_1^0, \ldots, \lambda_{n-1}^0$ of this polynomial equations system provides the points $P_i(\lambda_i^0)$ which determine a minimal curve $\beta(\lambda_1^0, \ldots, \lambda_{n-1}^0, t)$ and an approximation of the distance between Q_0 and Q_1 on the Bézier surface. If we cannot find an exact solution for $\lambda_1^0, \ldots, \lambda_{n-1}^0$, we just approximate them by using any resolution method for polynomial equations system.

IV. Examples

In this section we show the above procedure in three particular cases. We consider three pieces of regular surfaces, the cylinder, the sphere and the hyperbolic paraboloid, which are approximated by three Bézier surfaces. To approximate these regular surfaces the authors propose the normal approximation method described in [1].

We have chosen the cylinder and the sphere because their geodesics are well-known and easy to compute. Therefore, we can compare them with our approximated solutions. Concerning the hyperbolic paraboloid, we have chosen it because it is a polynomial surface and therefore, the necessary computations to produce the approximated minimal curve are easier than in other cases.

Example 1: Let $\chi(u, v)$ be an approximated Bézier surface of a piece of cylinder. We fix on $\chi(u, v)$ two points Q_0 and Q_1.

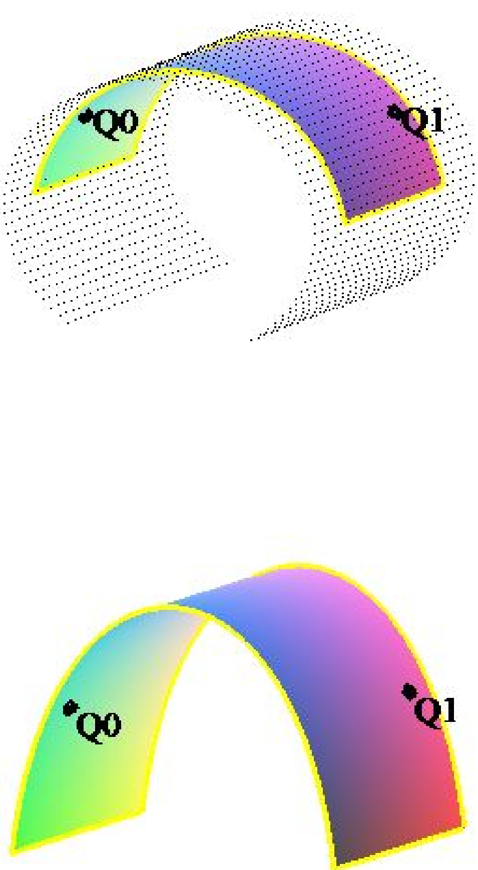


Figure 1. Approximated Bézier surface of cylinder, with Q_0 and Q_1.

We consider q_0 and q_1 in U such that $\chi(q_0) = Q_0$ and $\chi(q_1) = Q_1$. We make discrete orthogonal variations of the segment linking both points by taking two intermedite points $P_1(\lambda_1)$ and $P_2(\lambda_2)$. We construct the family $\alpha(\lambda_1, \lambda_2, t)$ of Bézier curves in U with control points q_0, $P_1(\lambda_1)$, $P_2(\lambda_2)$, q_1.

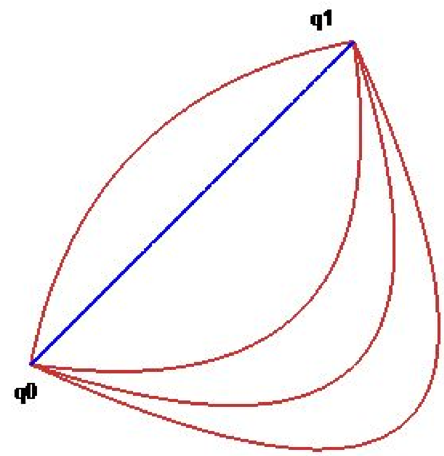


Figure 2. Discrete orthogonal variations of the segment between q_0 and q_1 in U.

We map the curves of the variation in U to the surface $\chi(u, v)$.

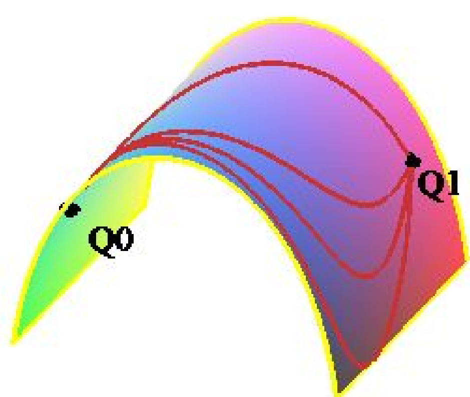


Figure 3. Variations of curves in $\chi(u, v)$.

We minimize the function $F(\lambda_1, \lambda_2, t)$ and we obtain a curve which provides the distance between Q_0 and Q_1 on $\chi(u, v)$.

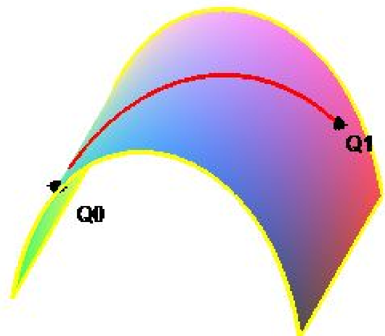


Figure 4. Minimal curve in $\chi(u,v)$.

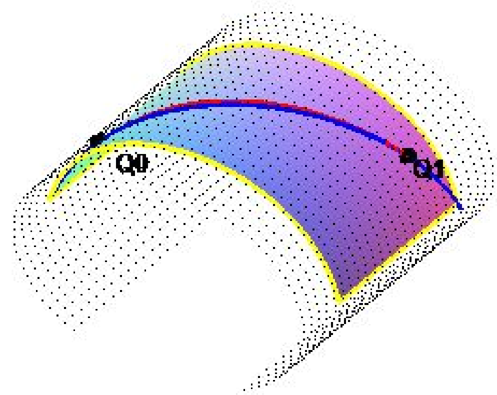
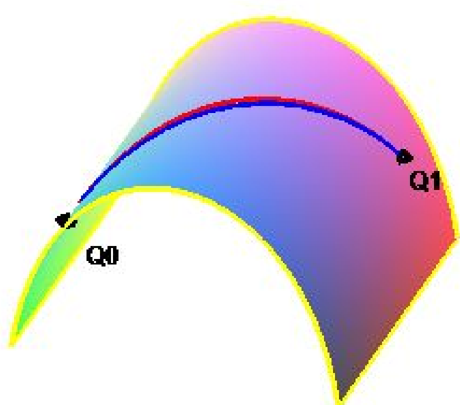
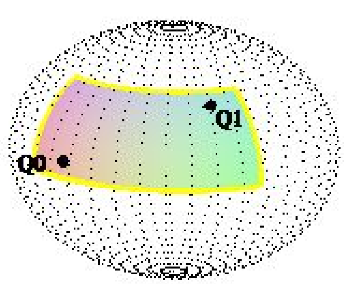


Figure 5. Minimal curve (in red) and image of the segment (in blue).

It is known that, in the real cylinder, the image of a segment in U is the minimal helix joining two points in the cylinder and this helix is the minimal geodesic between both points. If we consider the image of the segment between q_0 and q_1, we obtain the minimal helix linking Q_0 and Q_1. Therefore, the calculated minimal curve is an approximation of the minimal helix.



Example 2: Let $\chi(u,v)$ be an approximated Bézier surface of a piece of sphere and let Q_0 and Q_1 be two fixed points on it.



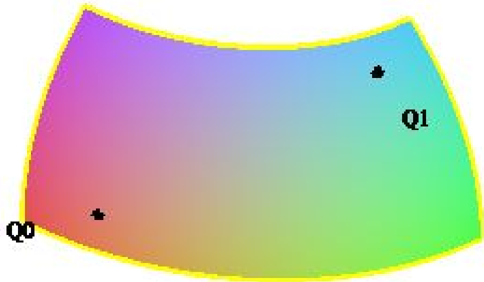


Figure 6. Approximated Bézier surface of sphere, with Q_0 and Q_1.

We consider q_0 and q_1 in U such that $\chi(q_0) = Q_0$ and $\chi(q_1) = Q_1$. We make discrete orthogonal variations of the segment linking both points by taking two intermedite points $P_1(\lambda_1)$ and $P_2(\lambda_2)$. We construct the family $\alpha(\lambda_1, \lambda_2, t)$ of Bézier curves in U with control points $q_0,\ P_1(\lambda_1),\ P_2(\lambda_2),\ q_1$. Look at Figure 2.

We map the curves of the variation in U to the surface $\chi(u, v)$.

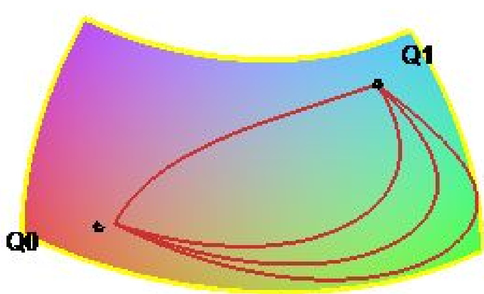
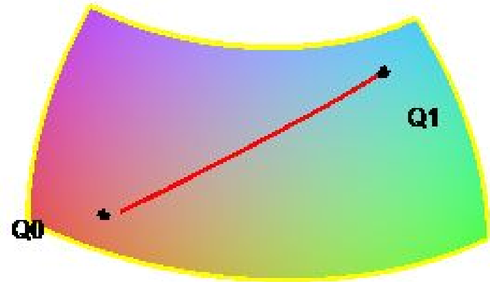


Figure 7. Variations of curves in $\chi(u, v)$.

We minimize the function $F(\lambda_1, \lambda_2, t)$ and we obtain a curve which provides the distance between Q_0 and Q_1 on $\chi(u, v)$.



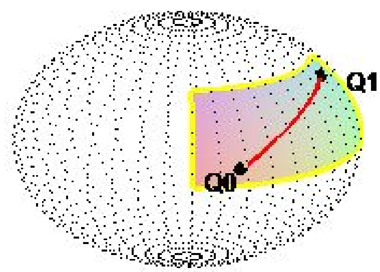


Figure 8. The minimal curve joining Q_0 and Q_1.

In the case of the sphere, we know that the geodesics are the great circles on it. Therefore, if we fix Q_0 and Q_1 on the approximated Bézier surface of the sphere, the intersection curve of the surface and the plane through Q_0, Q_1 and the origin, can supply us an approximation of the geodecic linking Q_0 and Q_1 on the sphere which provides the minimal distance between both points. Next, in the following figure, we compare graphically the obtained minimal curve and the approximated arc of great circle linking Q_0 and Q_1.

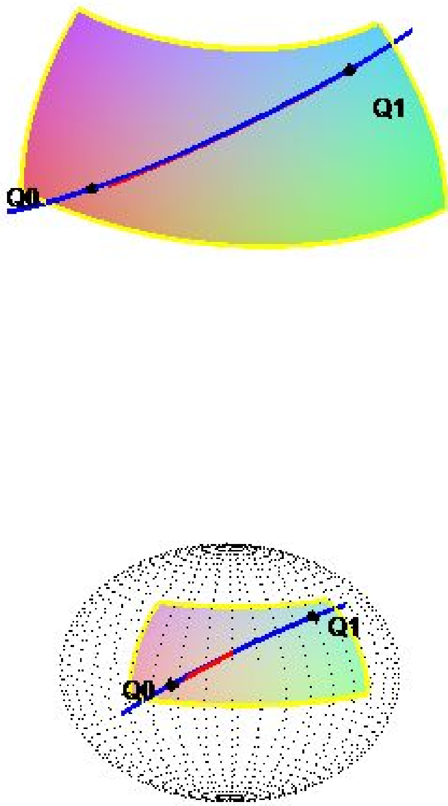


Figure 9. Comparison between the minimal curve (in red) and the great circle through Q_0 and Q_1 (in blue).

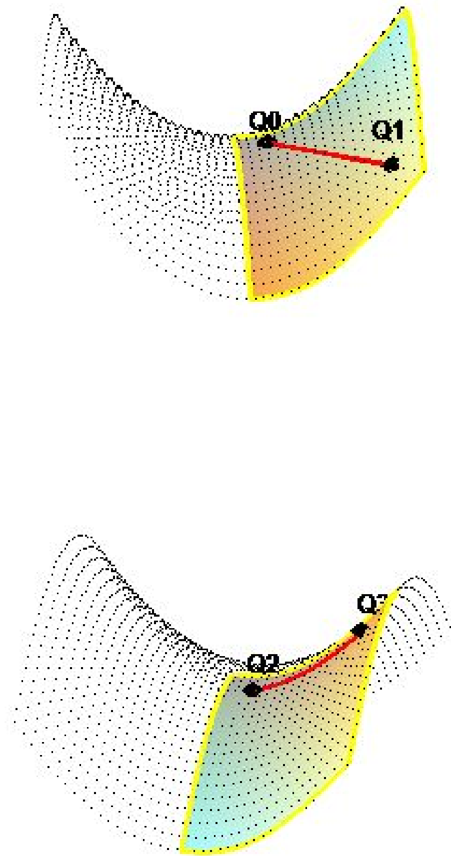


Figure 10. Minimal curves linking Q_0-Q_1 and Q_2-Q_3

Example 3: Let $\chi(u,v)$ be an approximated Bézier surface of a piece of hyperbolic paraboloid. We fix on $\chi(u,v)$ two couples of points: Q_0-Q_1 and Q_2-Q_3. We directly offer the final result in Figure 10. Let us point out that, in this case, the solution of the polynomial equations system in Step 4 is exact (not an approximated one).

V. Conclusion

The problem of computing the minimal curve between two points on an arbitrary surface is solved polynomially. If we approximate the surface by a Bézier surface, we can obtain discrete orthogonal variations by planar Bézier curves and we can minimize the length of the planar variational curves on the Bézier surface. We have, by way of examples, shown how this is feasible particularly for the Bézier approximation of the cylinder, the sphere and the hyperbolic paraboloid.

VI. Acknowledgement

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